Tuta Lecture 5

Recall from last time

$$\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d \quad / \mathrm{Spa}(\overline{\mathbb{F}_q})$$

$\mathrm{Pic}(S) = \{\text{line bundles on } X_S\}$ if $S \in \mathrm{Perf}_{\overline{\mathbb{F}_q}}$

$$\mathrm{Pic}^d \xrightarrow{\sim} [\mathrm{Spa}(\overline{\mathbb{F}_q}) / \underline{E}^\times]$$
$$\mathcal{L} \longmapsto \mathrm{Isom}(\mathcal{O}(d), \mathcal{L})$$

Div^d = pro-étale sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}_q}}$, $S \mapsto$ degree d effective Cartier divisors on X_S

$$\underbrace{\Sigma^d}_{\text{sum of } d \text{ degree 1 Cartier divisors}} : (\mathrm{Div}^1)^d \twoheadrightarrow \mathrm{Div}^d$$

quasi-proétale surjective

induces $(\mathrm{Div}^1)^d / \sigma_d \xrightarrow{\sim} \mathrm{Div}^d$

pro-étale quotient

$\overline{\mathbb{Q}}_\ell$ - local systems $/ \mathrm{Div}^1 \simeq \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$

$\chi : W_E \to \overline{\mathbb{Q}}_\ell^\times \rightsquigarrow \mathcal{E} =$ rk. 1 $\overline{\mathbb{Q}}_\ell$-local system on Div^1

$$\mathcal{E}^{(d)} := \left(\Sigma^d_* \mathcal{E}^{\boxtimes d} \right)^{\mathfrak{S}_d}$$

$=$ rk. 1 $\overline{\mathbb{Q}}_\ell$ - local system on Div^d.

The Abel Jacobi morphism :

$$\begin{aligned} AJ^d : \mathrm{Div}^d &\longrightarrow \mathrm{Pic}^d \\ D &\longmapsto \mathcal{O}(D) \end{aligned}$$

Def: $\mathbb{B}$ = sheaf of rings on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ defined by

$\circlearrowleft$
φ $\qquad \mathbb{B}(S) = \mathcal{O}(Y_S)$

$\mathbb{B}^{\varphi = \pi^d}$ = sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ of global sections of $\mathcal{O}(d)$

$$B(S)^{\varphi=\pi^d} = H^0(X_S, \mathcal{O}(d))$$

$B^{\varphi=\pi^d}$ = "absolute Banach Colmez space"

<u>not a diamond</u> (see later when $E = \overline{\mathbb{F}}_q((\pi))$)

but $B^{\varphi=\pi^d}$ relatively representable in diamonds (B.C. spaces)

$$\downarrow$$

$\mathrm{Spa}(\overline{\mathbb{F}}_q)$ = final object of the pro-étale topos, not representable

i.e. $\forall S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, $B_S^{\varphi=\pi^d}$ = diamond.

<u>Recall</u>: $\mathrm{Div}^d(S) = \{(\mathcal{L},u) \mid \mathcal{L}$ line bundle $/X_S$, $u \in H^0(X_S,\mathcal{L})$, $\forall s \in S$, $u_{|X_{k(s),k(s)^+}} \neq 0$, $\mathcal{L}$ deg. d fiberwise $/S\}/\sim$

$\mathcal{L}$: pro-étale locally on S $\simeq \mathcal{O}(d)$

$$\Downarrow$$

$$\mathrm{Div}^d = \underbrace{B^{\varphi=\pi^d} \setminus \{0\} / \underline{E}^\times}_{\text{proper spatial diamond } / \mathrm{Spa}(\overline{\mathbb{F}}_q)}$$

$$AJ^d : \mathbb{B}^{\varphi=\pi^d} \setminus \{0\} / \underline{E}^\times \longrightarrow \left[\mathrm{Spa}(\overline{\mathbb{F}_q}) / \underline{E}^\times \right] = \mathrm{Pic}^d$$

$\Rightarrow AJ^d =$ pro-étale locally trivial fibration in $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$

[__Th__: $\forall d \geq 2$ (resp. $d \geq 3$ if $E|\mathbb{Q}_p$) $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$ is simply connected (any finite étale cover has a section)]

__Consequence__: For $d \geq 3$, if $\mathcal{E} \leftrightarrow \chi : W_E \to \overline{\mathbb{Q}}_\ell^\times$

$\hookrightarrow$ on Div^1

and $\mathcal{E}^{(d)} := \left(\Sigma^d_* \mathcal{E}^{\boxtimes d} \right)^{\mathfrak{S}_d}$, $\mathcal{E}^{(d)}$ descends along AJ^d to a rk. 1 $\overline{\mathbb{Q}}_\ell$ - local system $\mathcal{F}^{(d)}$, $\mathcal{E}^{(d)} \overset{!}{=} (AJ^d)^* \mathcal{F}^{(d)}$, on Pic^d $\Rightarrow$ $\mathcal{E}$ descends along AJ^1 to a $\overline{\mathbb{Q}}_\ell$ - loc. sys. $/\mathrm{Pic}^1$

$\llcorner$ argument using the group structure of Pic

Now, one can compute the following ($E = \mathbb{Q}_p$)

Prop: Let $\chi_{LT} : \mathrm{Gal}(\bar{E}|E) \to \mathcal{O}_E^\times$ be the Lubin-Tate character attached to a L.T. group $/\mathcal{O}_E$ associated to the choice of the uniformizing element π.

Then $\pi_1(AJ^1) : \pi_1(\mathrm{Div}^1) \longrightarrow \pi_1(\mathrm{Pic}^1)$

$$
\begin{tikzcd}
\pi_1(\mathrm{Div}^1) \arrow[r] \arrow[d, equal] & \pi_1(\mathrm{Pic}^1) \arrow[d, equal] & {[\cdot / \underline{E}^\times]} \arrow[l] \\
W_E \arrow[r] & E^\times & \\
\tau \arrow[r, mapsto] & \chi_{LT}(\tau)\, \pi^{-v(\tau)} &
\end{tikzcd}
$$

where $\tau \bmod \pi = \mathrm{Frob}_q^{v(\tau)}$.

Thus if $\chi : W_E \to \bar{\mathbb{Q}}_\ell^\times$, $\mathcal{E}$ descends along AJ^1

$$\Updownarrow$$

χ factorizes through $W_E \to E^\times$.

$\Rightarrow$ local Kronecker_Weber theorem: $E^{ab} = E^{un}$(torsion points of Lubin-Tate)

Proof of the simple connectedness when $E = \mathbb{F}_q((\pi))$

Recall: $Y_S = \mathbb{D}^*_S \subset \mathbb{A}^1_S$

$$\mathbb{D}^*_S = \{0 < |\pi| < 1\}$$

$\Rightarrow$ $\forall S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid $/\mathbb{F}_q$

$$B(S) = \left\{ \sum_{n \in \mathbb{Z}} x_n \pi^n \;\middle|\; x_n \in R \text{ and } \forall \rho \in]0,1[\; \lim_{|n| \to +\infty} \|x_n\| \rho^n = 0 \right\}$$

$\|\cdot\|$ = power multiplicative on R.

Using this one computes:

$$(R^{oo})^d \xrightarrow{\sim} B(S)^{\varphi = \pi^d}$$

$$(x_0, \dots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{b \in \mathbb{Z}} x_i^{q^{-b}} \pi^{bd+i}$$

④

$\Rightarrow \mathbb{B}^{\varphi=\pi^d}$ is represented by the perfect adic space

$\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]],\overline{\mathbb{F}}_q[[x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]]\right)$ = not perfectoid since not analytic.

$\Rightarrow$ ~~[illegible]~~ but $\forall S\in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, $\mathbb{B}_S^{\varphi=\pi^d}\simeq \mathring{\mathbb{B}}^d_{S'}$ = perfectoid space

i.e. $\mathbb{B}^{\varphi=\pi^d}$ = not perfectoid

$\downarrow$ relatively representable in perfectoid spaces.

$\mathrm{Spa}(\overline{\mathbb{F}}_q)$ = not perfectoid

* <u>But</u>: $\mathbb{B}^{\varphi=\pi^d}\setminus\{0\}\simeq \mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]],-\right)\setminus V(x_0,\dots,x_{d-1})$

$= \mathrm{Spa}(-)_a$ ← analytic points

$=$ perfectoid space $= \bigcup_{i=0}^{d-1} \underbrace{\{x_i\neq 0\}}_{\mathring{\mathbb{B}}^{d-1,1/p^\infty}_{\overline{\mathbb{F}}_q((x_i^{1/p^\infty}))}}$

= Union of d-perfectoid open balls over different perfectoid fields.

Rem: As a sheaf of $\underline{E}$-vector spaces

$$\mathbb{B}^{\varphi=\pi^d} = \varprojlim_{\times\pi} \underbrace{\mathcal{G}}_{\text{Drinfeld-module}} = \text{universal cover of } \mathcal{G}$$

$$= \text{formal } E\text{-vector space}$$

where $\mathcal{G} = \widehat{\mathbb{G}}_a^{\,d} \circlearrowleft \mathbb{F}_q[[\pi]]$

$$\pi \cdot (x_0, \dots, x_{d-1}) = (\mathrm{Frob}_q(x_{d-1}), x_0, \dots, x_{d-2})$$

Proof of the simple connectedness

Elkik approximation + $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$ q.c.q.s.

$$\Downarrow$$

finite étale

$$2\text{-}\varinjlim_{n\geq 1} \overbrace{\text{F.ét}}\Big/ \mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^n}, \dots, x_{d-1}^{1/p^n}]]\right) \setminus V(x_0, \dots, x_{d-1})$$

$$\downarrow \simeq$$

$$\text{F.ét.}\Big/ \mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right) \setminus V(x_0, \dots, x_{d-1})$$

It thus suffices to prove that

$$\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0, \dots, x_{d-1}]], \overline{\mathbb{F}}_q[[x_0, \dots, x_{d-1}]]\right) \setminus V(x_0, \dots, x_{d-1})$$

is simply connected.

<u>Prop</u> (GAGA): A = I-adic noetherian ring. Then

$$\text{F.ét.}\Big/ \mathrm{Spec}(A) \setminus V(I) \xrightarrow{\sim} \text{F.ét.}\Big/ \mathrm{Spa}(A, A) \setminus V(I)$$

$\Rightarrow$ Suffices to proove that $\mathrm{Spec}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right)\setminus V(x_0,\dots,x_{d-1})$ is simply connected for $d\geq 2$.

$\longrightarrow$ Zariski-Nagata purity for

$$\mathrm{Spec}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right)\setminus V(x_i)_i \hookrightarrow \mathrm{Spec}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right)$$

+ Hensel $\Rightarrow$ $\mathrm{Spec}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right)$ is simply connected. $\square$

Rem: When $E|\mathbb{Q}_p$ this gives an indication on what to do: One checks first that it suffices to prove purity for

$$\mathbb{B}^{\varphi=\pi^d}\setminus\{0\} \hookrightarrow \mathbb{B}^{\varphi=\pi^d}$$

Proof of purity is very complicated.